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Solitons and Breathers of Electromagnetic Wave in Superlattices

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We present that the vector potential of electromagnetic wave propagating in superlattices obeys the sine-Gordon equation. The soliton and breather solutions are given. The well-known soliton and breather solutions of the sine-Gordon equation provide a theoretical description of the vector potential of electromagnetic wave propagating in superlattices.

KEY WORDS: superlattice; soliton; breather; sine-Gordon equation.

Nonlinear wave propagation through inhomogeneous media has attracted considerable attention (Broderich and Martijn, 1995; Kivshar, 1993) in recent years. One of the simplest and physically relevant models of an inhomogeneous continuous medium is that with periodic variations on its parameters. In these systems, superlattices have been extensively studied by both experimental (Grahn *et al.*, 1991; Kwok *et al.*, 1995) and theoretical techniques (Bulashenko *et al.*, 1996; Kastrup *et al.*, 1997). Superlattices are fascinating because the structures exhibit collective properties not shares by either constituent, and these characteristics can be controlled through variation of the structural parameters.

Soliton phenomena in superlattices have been attracting considerable attention. We have shown that the negative differential conductivity in weakly coupled narrow-miniband semiconductors results in the formation of electric-field domains, and the envelope of its wave function is governed by the nonlinear Schr *ö* dinger equation (NLS) which has soliton solutions (Martijn de Sterke and Sipe, 1988; Tian *et al.*, 2001; Tian and Wu, 1999).

In the present paper we will show that the vector potential of electromagnetic wave propagating in superlattices obeys the sine-Gordon equation (SG). Based on the inverse scattering method, the soliton and breather solutions are given, which

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provide a theoretical description of the vector potential of electromagnetic wave propagating in superlattices.

Except for very thin barriers, the tight binding model provides an adequate description for the superlattice miniband energy dispersion $E(\vec{p})$

$$E(\vec{p}) = \frac{p_{xy}^2}{2m} + \frac{W}{2}\left(1 - \cos\frac{p_z d}{\hbar}\right) \tag{1}$$

where W is the miniband width of superlattice in z direction, d is the superlattice period.

In the presence of electromagnetic wave propagating in the superlattice in z direction, we make the Peierls substitution (Hofstadter, 1976; Tian, 1998): we replace $\vec{p}$ by $\vec{P} + e\vec{A}(t)$ in the dispersion relation $E(\vec{p})$, where $\vec{P}$ is the momentum canonical to $\vec{r}$, $\vec{A}(t)$ is the vector potential of the electromagnetic wave, e is electron charge.

From the resulted dispersion, the current density in z direction can be easily obtained

$$\begin{aligned} j_z(t) &= -e\sum_{P_z} v_z(P_z + eA_z) \\ &= -e\frac{Wd}{2\hbar}\sum_{P_z}\sin\frac{(P_z + eA_z(t))d}{\hbar} \\ &= -e\frac{Wd}{2\hbar}\sum_{P_z}\left(\sin\frac{P_z d}{\hbar}\cos\frac{eA_z(t)d}{\hbar} + \cos\frac{P_z d}{\hbar}\sin\frac{eA_z(t)d}{\hbar}\right) \end{aligned} \tag{2}$$

in which the relation $v_z = \frac{\partial}{\partial P_z}E(\vec{P} + e\vec{A}(t))$ is used.

It is known that the equilibrium distribution in momentum space of superlattice electrons is centrosymmetric without external field and $j_z = 0$. Under an applied electric field, the distribution function in momentum space will be unsymmetrical. Now, we investigate the electromagnetic wave propagating in superlattice with a weak applied electric field. The following two properties are assumed:

1. Under a weak applied electric field, the momentum distribution is not far from centrosymmetric state (Tian, 1998; Tian and Ma, 1998). We approximately have $\sum_{P_z}\sin\frac{P_z d}{\hbar} = 0$. Then the current density Eq. (2) can be written as

$$j_z(t) = -e\frac{Wd}{2\hbar}\sin\frac{eA_z(t)d}{\hbar}\sum_{P_z}\cos\frac{P_z d}{\hbar} \tag{3}$$

2. Considering that the wave length of electromagnetic wave is much larger than the superlattice period d, the superlattice can be treated as a

homogeneous medium for the electromagnetic wave. Thus

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{j} \tag{4}$$

where c is light velocity in the superlattice and μ is susceptibility. In the z direction, it is

$$\nabla^2 A_z - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = -\mu j_z(t) \tag{5}$$

Inserting Eq. (3) into (5) gives

$$\frac{\partial^2 A_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = e\mu \frac{Wd}{2\hbar} \sin \frac{eA_z(t)d}{\hbar} \sum_{P_z} \cos \frac{P_z d}{\hbar} \tag{6}$$

This is a sine-Gordon equation.

To simplify notations, let

$$\phi(t) = \frac{eA_z(t)d}{\hbar} \tag{7a}$$

$$\frac{1}{g^2} = e^2\mu \frac{Wd^2}{2\hbar^2} \sum_{P_z} \cos \frac{P_z d}{\hbar} \tag{7b}$$

They bring Eq. (6) to

$$\frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{1}{g^2} \sin\phi \tag{8}$$

With the new independent variables ξ and τ

$$\xi = \frac{1}{2g}(z + ct) \tag{9a}$$

$$\tau = \frac{1}{2g}(z - ct) \tag{9b}$$

It is straightforward to obtain the standard sine-Gordon equation (SG equation)

$$\theta_{\xi\tau} = \sin\theta \tag{10}$$

where $\theta(\xi, \tau) = \phi(z, t)$, and a derivative is indicated by a subscript.

It is well known that the SG equation has soliton and breather solutions. Using the inverse scattering method, the Lax pair corresponding to the SG equation (10) are (Lamb, 1980)

$$L(\lambda) = -i\lambda\sigma_3 + U \tag{11a}$$

$$M(\lambda) = \frac{i}{4\lambda} V \tag{11b}$$

where

$$U = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}, \qquad u = -\frac{1}{2}\theta_\xi \tag{12}$$

$$V = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{13}$$

and σ_3 is the Fermi matrix, λ is a parameter determined by the boundary conditions.

After a lengthy procedure, the solution of $u(\xi, \tau)$ is written in the form

$$u(\xi, \tau) = 2\frac{\partial}{\partial\xi} \arctan \frac{\mathrm{Im}\ \det(I - iM)}{\mathrm{Re}\ \det(I - iM)} \tag{14}$$

where I is a unit matrix.

Because $u = -\frac{1}{2}\theta_\xi$, we immediately find that the solutions will be

$$\theta = -4 \arctan \frac{\mathrm{Im}\ \det(I - iM)}{\mathrm{Re}\ \det(I - iM)} \tag{15}$$

For the simplest case of single soliton, the matrix M reduces to the scalar

$$M = e^{\Theta_1} \tag{16}$$

where

$$\Theta_1 = 2\nu_1(\xi - \xi_1) + \frac{\tau}{2\nu_1} \tag{17}$$

The subscript "1" indicates single soliton. Here $\nu_1 = \mathrm{Im}\,\lambda_1$, $\lambda_1 = \mu_1 + i\nu_1$, ξ_1 is a constant. They are determined by the boundary conditions and initial conditions. Then the single-soliton solution for the SG equation is

$$\theta_1 = 4 \arctan e^{\Theta_1} \tag{18}$$

For the breather solution, in which the oscillations and the envelope move at different velocities, we obtain

$$\frac{\mathrm{Im}\ \det(I - iM)}{\mathrm{Re}\ \det(I - iM)} = \frac{\nu_1}{\mu_1}\frac{\sin\Phi_\mathrm{b}}{\cosh\Theta_\mathrm{b}} \tag{19}$$

where

$$\Theta_\mathrm{b} = 2\nu_1(\xi - \xi_1) + \frac{2\nu_1}{4\left(\mu_1^2 + \nu_1^2\right)}\tau \tag{20}$$

$$\Phi_\mathrm{b} = 2\mu_1\xi - \frac{2\mu_1}{4\left(\mu_1^2 + \nu_1^2\right)}\tau + \varphi \tag{21}$$

φ is a real constant. The subscript "b" indicates breather solution. Finally, the breather solution is then

$$\Theta_\mathrm{b} = -4 \arctan \frac{\nu_1\ \sin\Phi_\mathrm{b}}{\mu_1\ \cosh\Theta_\mathrm{b}} \tag{22}$$

In conclusion, we present a new soliton phenomenon in this paper. We show that the vector potential of electromagnetic wave propagating in superlattices obeys the sine-Gordon equation. Based on the inverse scattering method, the soliton and breather solutions are given, which provide a theoretical description of the vector potential of electromagnetic wave propagating in superlattices.

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